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Solution will be uploaded after the tutorial on Wednesday.

Recall

We shall prove (iii) $\implies$ (i) in this tutorial. Then we finish the proof.

Theorem 1.3 Let (X, d) be a metric space and $K \subset X$. Then the following are equivalent:

(i) K is compact

(ii) Every sequence in K has a convergent subsequence which converges in K

(iii) K is complete and totally bounded

Last week you learned:

Theorem 4.2 (Ascoli's Theorem) Consider $C(\overline{G})$ where G is bounded, open in $\mathbb{R}^n$. A set $\mathcal{E}$ in $C(\overline{G})$ is precompact if it is bounded and equicontinuous.

Theorem 4.4 (Arzelà's Theorem) Every precompact set in $C(\overline{G})$ must be bounded and equicontinuous.

They are usually put together and called Arzelà-Ascoli's Theorem. It is one of the most essential and fundamental theorem in analysis.

In addition to the given statement, if $\mathcal{E}$ is closed, then Arzelà-Ascoli can be written as $\mathcal{E}$ compact $\iff$ bounded and equicontinuous. (**Check!**)

Note that, in different context, there is a slightly different version of the Arzelà-Ascoli's Theorem. But they are more or less the same. We state the version of Arzelà-Ascoli's Theorem that we will be using later:

Arzelà-Ascoli's Theorem Let $I = [a, b]$ be a compact interval. Let $\mathbf{x}_n(t) : I \to \mathbb{R}^d$ be a sequence of functions such that

(i) there exists a constant $M > 0$ such that $|x_n(t)| \leq M$ for any n and $t \in I$, i.e., $\{\mathbf{x}_n(t)\}$ is uniformly bounded; and

(ii) the sequence $\{\mathbf{x}_n(t)\}$ is equicontinuous in I.

Then there exists a subsequence $\{\mathbf{x}_{n_k}(t)\}$ that converges uniformly on I to a limit function $\mathbf{y}(t)$ as $n \to \infty$.

Proof of (iii) $\implies$ (i)

Idea: Proof by contradiction.
Goal: **For any** open covering of X, which is complete and totally bounded, we want to show the existence of **a** finite subcovering such that it covers X.

Proof:
Suppose **there exists** an open cover, $\{U_\alpha\}_{\alpha \in I}$, of X such that it does not contain **any** finite subcover from $\{U_\alpha\}_{\alpha \in I}$ that covers X.

Since X is totally bounded, then it can be covered by a finite ε-net. In particular, we may take $\varepsilon = 1$ for simplicity. If all these balls can be covered by finitely many U_α, then it contradicts our assumption. If not, then there must be a ball, say, $B(x_0, 1)$, that cannot be covered by finitely many U_α.

Since X is totally bounded, then $B(x_0, 1)$ is also totally bounded. In particular, we may cover $B(x_0, 1)$ by a finite ε-net, in which we choose $\varepsilon = \frac{1}{2}$ this time. Then one can observe that the centers of each $\frac{1}{2}$-ball must be at most $1 + \frac{1}{2}$ away from x_0. Otherwise, the ball will not cover $B(x_0, 1)$. If $B(x_0, 1)$ can be covered by finitely many U_α, then there is a contradiction. If not, then there is a ball $B(x_1, \frac{1}{2})$ such that it cannot be covered by finitely many U_α. Moreover, the above discussion implies

$$d(x_0, x_1) \leq 1 + \frac{1}{2}$$

We are starting to see a loop here. Since X is totally bounded, then $B(x_1, \frac{1}{2})$ is also totally bounded. Then with a similar argument, one can see that

$$d(x_1, x_2) \leq \frac{1}{2} + \frac{1}{4}$$

With the above construction, we obtain a sequence of point $x_0, x_1, ...$ in X such that each ball $B(x_n, 2^{-n})$ cannot be covered by finitely many U_α and that

$$d(x_n, x_{n+1}) \leq 2^{-n} + 2^{-n-1}$$

for all $n = 1, 2, ...$ From this, we see that $\{x_n\}$ is Cauchy as

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

Together with the fact that X is complete, then it has a limit, say, x, that lies in X.

Now that we have obtained a convergent sequence, and that $\{U_\alpha\}_{\alpha \in I}$ is an open cover of X. Then the limit x must lie in at least one of the U_α's. Moreover, U_α is open, and $x \in U_\alpha$, so there exists $r > 0$ such that $B(x, r) \subset U_\alpha$. Also, the convergence of x_n means that we can find an n such that

$$d(x_n, x) < \frac{r}{2}$$

for which $2^{-n} < \frac{r}{2}$. In particular, this implies $B(x_n, 2^{-n}) \subset B(x, r) \subset U_\alpha$. Contradiction.

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Exercise 1

Source: Previous HW Problem from MATH4051@HKUST by Prof Frederick Fong

Suppose $\mathbf{F}_k = (F_k^1, ..., F_k^d) : \mathbb{R}^d \to \mathbb{R}^d$ is a sequence of C^2-vector fields such that there exists a constant $M > 0$ such that

$$\sum_{i=1}^{d} |F_k^i(\mathbf{x})| + \sum_{i,j=1}^{d} \left| \frac{\partial F_k^i}{\partial x_j}(\mathbf{x}) \right| + \sum_{i,j,l=1}^{d} \left| \frac{\partial^2 F_k^i}{\partial x_j \partial x_l}(\mathbf{x}) \right| \leq M \tag{0.1}$$

for any $\mathbf{x} \in \mathbb{R}^d$ and $k \in \mathbb{N}$.

Show that there exists a subsequence of $\{\mathbf{F}_k\}_{k=1}^{\infty}$ which converges uniformly on $\mathbb{R}^d$ to a limit $\mathbf{F}_\infty : \mathbb{R}^d \to \mathbb{R}^d$.

Solution:

Equation (0.1) tells is that $\{\mathbf{F}_k\}$ and $\|D\mathbf{F}_k\|$ are both uniformly bounded on $\mathbb{R}^d$. Proposition 4.1 then implies that $\{\mathbf{F}_k\}_{k=1}^{\infty}$ is equicontinuous on $\mathbb{R}^d$. However, if we are to apply the Arzelà-Ascoli's Theorem, we need to apply it over a compact (or closed and bounded, but they are the same over Euclidean space) domain. We do it as follows:

- Consider the ball $\overline{B_1(\mathbf{0})}$, which is compact, then Arzelà-Ascoli's Theorem implies that there exists a subsequence $\{\mathbf{F}_{1,k}\}_{k=1}^{\infty}$ such that it converges uniformly to a limit function $\mathbf{F}_\infty$ on $\overline{B_1(\mathbf{0})}$.
- Then consider the ball $\overline{B_2(\mathbf{0})}$ and the subsequence $\{\mathbf{F}_{1,k}\}_{k=1}^{\infty}$, which is also equicontinuous and uniformly bounded on $\overline{B_2(\mathbf{0})}$. Then, Arzelà-Ascoli's Theorem implies there exists a subsequence $\{\mathbf{F}_{2,k}\}_{k=1}^{\infty} \subset \{\mathbf{F}_{1,k}\}_{k=1}^{\infty}$ such that it converges uniformly to a limit function on $\overline{B_2(\mathbf{0})}$. By the uniqueness of limit, such a subsequence must converges to a limit function in which it coincides with $\mathbf{F}_\infty$ on $\overline{B_1(\mathbf{0})}$. We may denote the limit function of $\{\mathbf{F}_{2,k}\}_{k=1}^{\infty}$ on $\overline{B_2(\mathbf{0})}$ by $\mathbf{F}_\infty$ as well.
- Inductively: There exists subsequences

  $$\{\mathbf{F}_{1,k}\}_{k=1}^{\infty} \supset \{\mathbf{F}_{2,k}\}_{k=1}^{\infty} \supset \{\mathbf{F}_{3,k}\}_{k=1}^{\infty} \supset \cdots$$

  such that for each $n \in \mathbb{N}$, the sequence $\{\mathbf{F}_{n,k}\}_{k=1}^{\infty}$ converges uniformly to a limit function $\mathbf{F}_\infty$ on $\overline{B_n(\mathbf{0})}$.

Next, we will use a diagonalization argument as in the proof of the Arzelà-Ascoli's Theorem.

Consider a diagonal sequence $\{\mathbf{F}_{k,k}\}_{k=1}^{\infty}$. We want to show that this is the subsequence as desired. For any compact set K, there exists $N > 0$ large enough so that $K \subset \overline{B_N(\mathbf{0})}$ (boundedness of K). Note that, for any $n \geq N$, the sequence $\{\mathbf{F}_{n,k}\}_{k=1}^{\infty} \subset \{\mathbf{F}_{N,k}\}_{k=1}^{\infty}$, and hence $\{\mathbf{F}_{n,n}\}_{n=N}^{\infty} \subset \{\mathbf{F}_{N,k}\}_{k=1}^{\infty}$. Since $\{\mathbf{F}_{N,k}\}_{k=1}^{\infty}$ converges uniformly to F_∞ on $\overline{B_N(\mathbf{0})} \supset K$ as $k \to \infty$, so $\{\mathbf{F}_{n,n}\}_{n=N}^{\infty}$ converges uniformly on K. Then, by adding finitely many terms to the sequence $\{\mathbf{F}_{n,n}\}_{n=N}^{\infty}$, the sequence $\{\mathbf{F}_{k,k}\}_{k=1}^{\infty}$ converges uniformly on K to $\mathbf{F}_\infty$ as $k \to \infty$.

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